Grade 9/10 Math Circles
October 25, 2023
Graph Theory - Solutions

Exercise Solutions

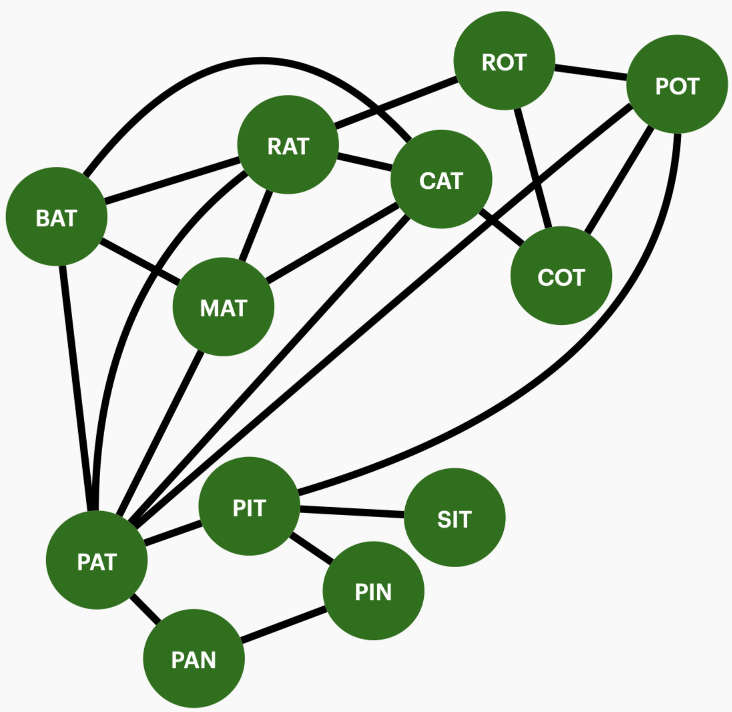
Exercise 1

Create a 'word graph', where the vertices are:

- Bat
- Cot
- Pan
- Pin
- Pot
- Rot
- Cat
- Mat
- Pat
- Pit
- Rat
- Sit

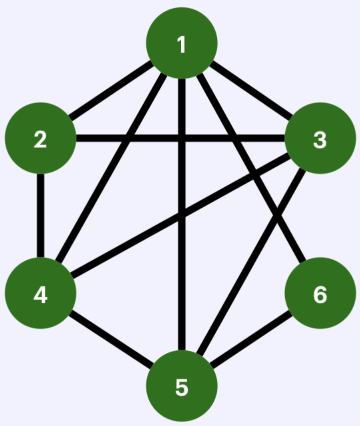
and two vertices are neighbours if they differ by only one letter.

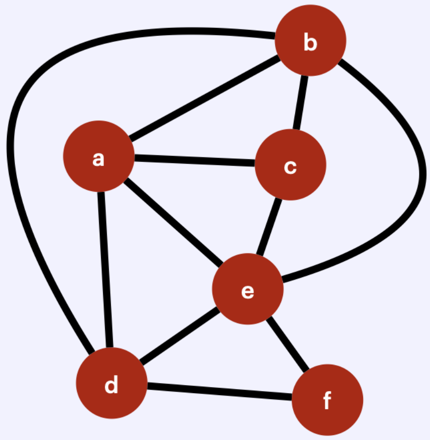
Exercise 1 Solution



Exercise 2

Show that the following two graphs are isomorphic





Exercise 2 Solution

We show that they are isomorphic by providing an isomorphism as follows: 1 = e, 2 = c, 3 = a, 4 = b, 5 = d and 6 = f.

Exercise 3

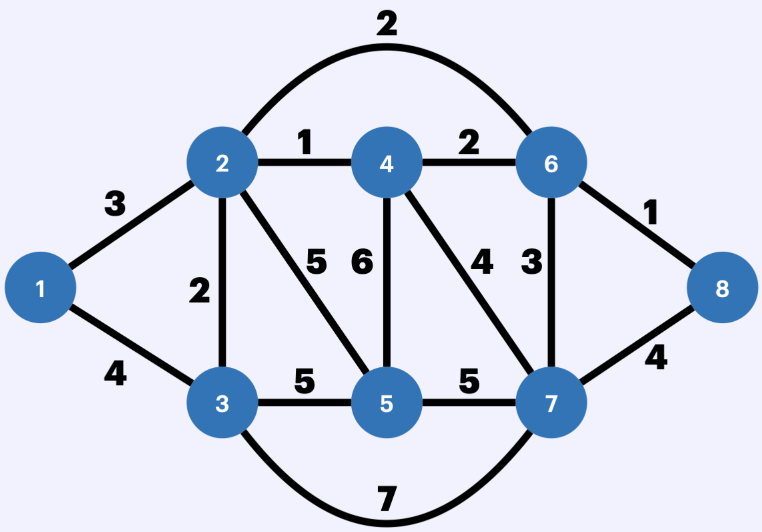
Find a graph with vertex set $\{a, b, c, d\}$ where $\deg(a) = 1$, $\deg(b) = 3$, $\deg(c) = 2$, $\deg(d) = 1$.

Exercise 3 Solution

There is no such graph! The sum of the degrees is $1 + 3 + 2 + 1 = 7$. By the Handshaking Lemma, this would mean that we have 3.5 edges, which is not valid.

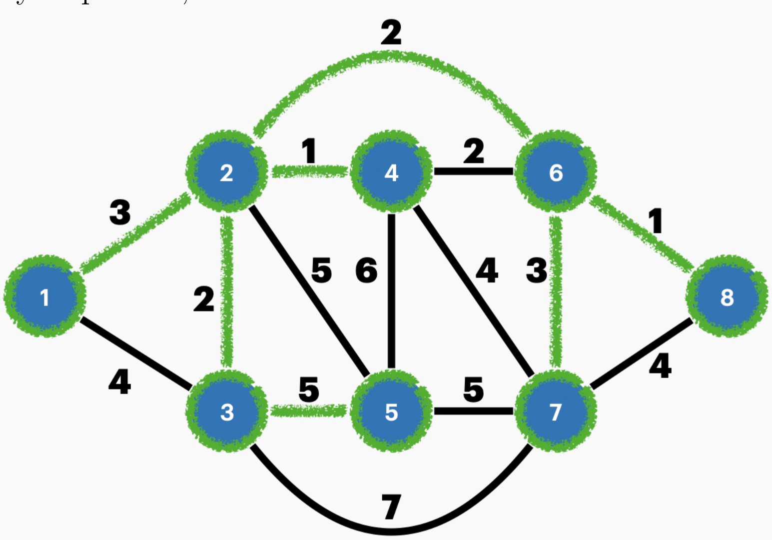
Exercise 4

Find an MST of the graph below:



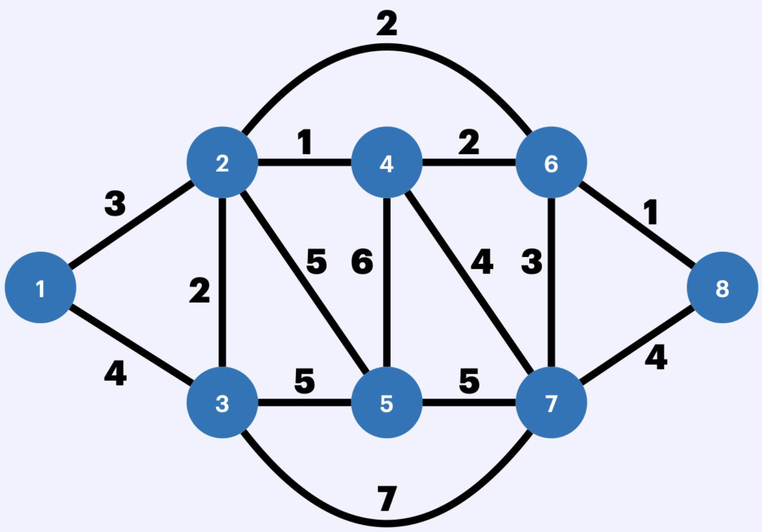
Exercise 5 Solution

By inspection, we find:



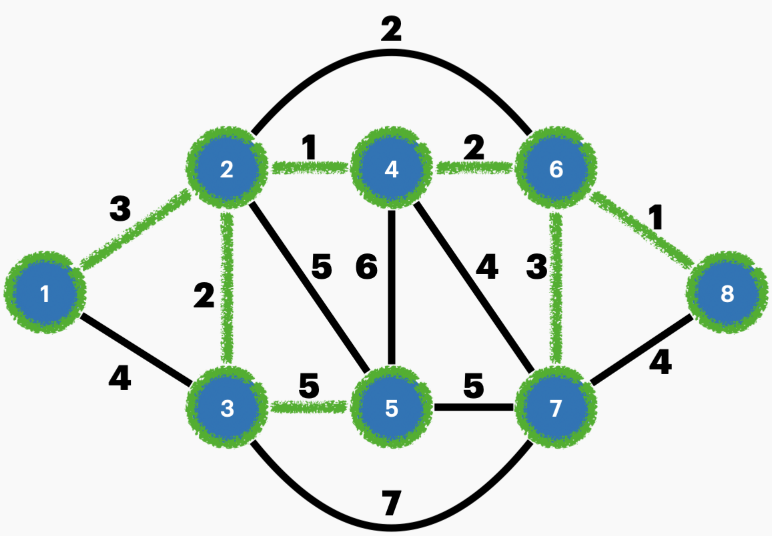
Exercise 5

Use Prim's Algorithm to find a different MST of the graph below:



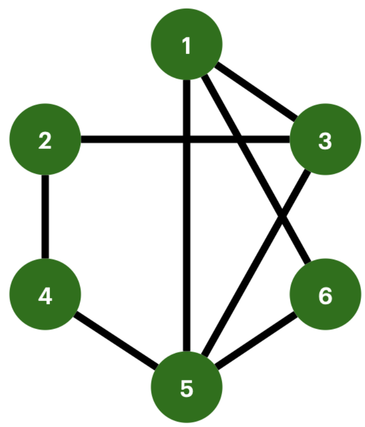
Exercise 5 Solution

According to Prim's Algorithm, we do the following: Starting from vertex 1, we add edges $\{1, 2\}$, $\{2, 4\}$, $\{2, 3\}$, $\{4, 6\}$, $\{6, 8\}$, $\{6, 7\}$ and finally $\{3, 5\}$. This gives an MST of weight $3+1+2+2+1+3+5 = 17$. The MST is given visually below:



Problem Set Solutions

Graph Basics



Graph A

1. Find the vertex and edge set of Graph A.

> *Solution*:
> Vertices $= \{1, 2, 3, 4, 5, 6\}$
> Edges $= \{\{1, 3\}, \{1, 5\}, \{1, 6\}, \{2, 3\}, \{2, 4\}, \{3, 5\}, \{4, 5\}, \{5, 6\}\}$

2. For Graph A:
   (a) Find the neighbours of vertex 1 and vertex 5
   (b) Find the degree of vertex 1 and vertex 5
   (c) What do you notice about the degree and the neighbours of a given vertex? Why is this the case?

> *Solution*:
> (a) The neighbours of vertex 1 are 3, 5 and 6. The neighbours of vertex 5 are 1, 3, 4 and 6.
> (b) $\deg(1) = 3$ and $\deg(5) = 4$
> (c) The number of neighbours is equal to the degree of a vertex. This is because the degree is calculating the number of edges connected to a vertex. Each edge is connected to a different neighbour.

3. For Graph A:
   (a) Find a walk from vertex 1 to 4
   (b) Find a path from vertex 2 to 5
   (c) Find a cycle
   (d) Find a spanning tree

Solution: Answers may vary. Here are some example answers.

(a) 1, 5, 3, 1, 6, 5, 4

(b) 2, 3, 5

(c) 2, 4, 5, 3, 2

(d) Include edges: $\{\{1,5\},\{1,3\},\{1,6\},\{2,3\},\{2,4\}\}$

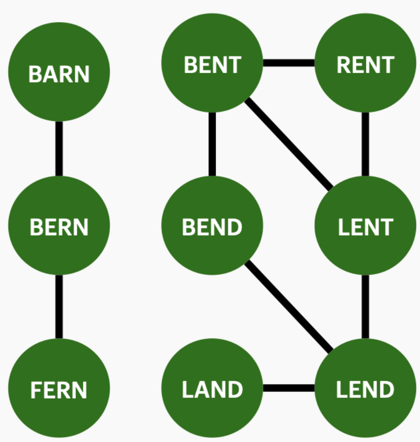
4. Describe a graph (with vertex and edge sets) that, when drawn, can be in the shape of something fun!

Solution: Be creative!

Word Graphs

1. Create a word graph using the following words:
   BARN, BEND, BENT, BERN, FERN, LAND, LEND, LENT, RENT

Solution:



2. Find a potential path in a word graph from MATH to TEAM.
   Hint: Try passing through the word PEAS along the way!

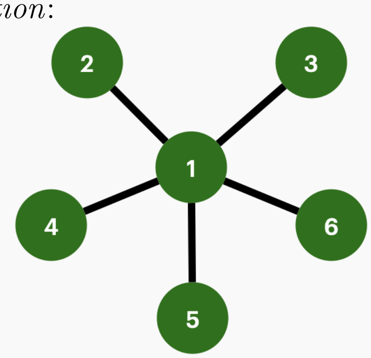
> *Solution*: math - path - pats - pets - peas - teas - team

3. What would the word graph of {a, b, c, d, e, . . ., v, w, x, y, z}) look like?

> *Solution*: It would be a graph with 26 vertices, where every vertex is connected to each other vertex. (We can get all letters from another by simply changing one letter)

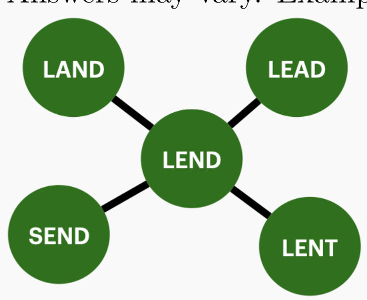
4. A star is a graph each vertex (aside from the 'centre') has degree 1 and is connected by an edge to the 'centre'. An example is given (Graph B).
   (a) Call a star with k non-'centre' vertices S_k. Draw the S_5 graph.
   (b) Create a word graph that has the shape of S_4.
   (c) **Challenge:** What properties must the words satisfy in order to create an S_k word graph?

Solution:



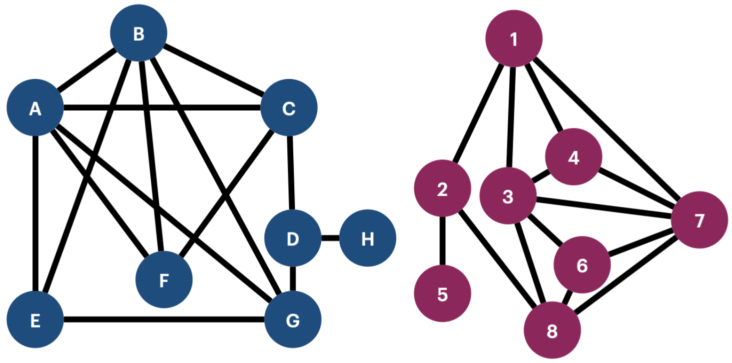
(a)

(b) Answers may vary. Example:

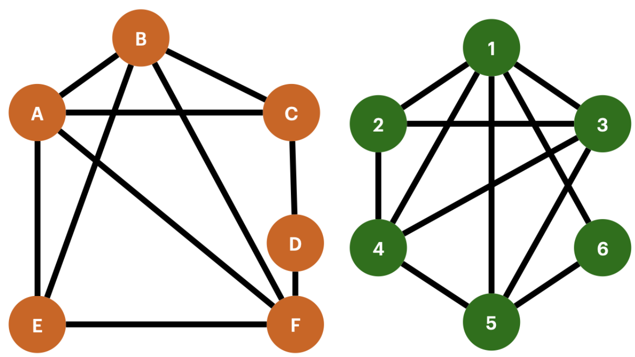


(c) In order to create a star graph, we must change a different letter from the centre vertex each time (otherwise we wouldn't have a star). This also means that the words must be at least length k to create an S_k graph.

Isomorphic Graphs



Pair A



Pair B

1. Prove the graphs in Pair A are isomorphic by providing an isomorphism between them.

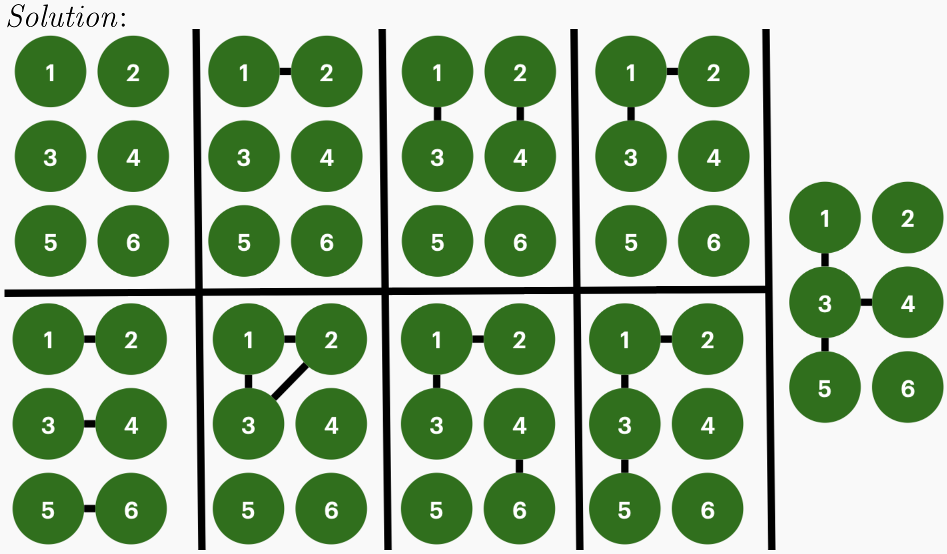
Solution: There may be correct multiple answers. One solution: 1 = C, 2 = D, 3 = B, 4 = F, 5 = H, 6 = E, 7 = A, 8 = G.

2. **Challenge:** Prove that the graphs in Pair B are not isomorphic.

Solution: Note that the graph on the left has 10 edges whereas the graph on the right has 11. This means that they cannot be isomorphic. Also, no vertex in the graph on the left has degree 5, but deg(1) = 5.

3. Draw all non-isomorphic graphs which have 6 vertices and less than 4 edges.
   Hint: There should be 9 such graphs.

Solution:



Handshaking Lemma

1. Confirm that your graph from Graph Basics Q4 satisfies the Handshaking Lemma.

Solution:
Answers vary. Confirm that the sum of the degrees of the vertices is twice the number of edges.

2. A graph is k-regular if each vertex has degree k. Find the number of edges in a 3-regular graph with 10 vertices.

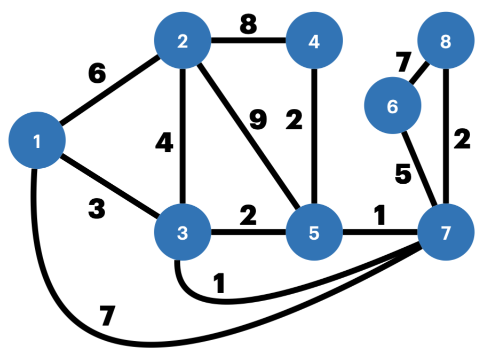
Solution:
Sum of degrees $= 30$. Hence, by the Handshaking Lemma, the number of edges is $30/2 = 15$.

3. Find the number of vertices in a 4-regular graph with 72 edges.

> *Solution*:
> The sum of the degrees is 4 times the number of vertices. Let v represent the number of vertices. Hence, by the Handshaking Lemma, $4v = 2 * 72 \implies v = 144/4 = 36$.

Prim's Algorithm

1. Find an MST of Graph C using Prim's Algorithm.



Graph C

> *Solution*:
> According to Prim's Algorithm, we do the following: Starting from vertex 1, we add edges $\{1,3\}$, $\{3,7\}$, $\{5,7\}$, $\{7,8\}$, $\{4,5\}$, $\{2,3\}$ and finally $\{6,7\}$. This gives an MST of weight 3+1+1+2+2+4+5 = 18. The MST is given visually below: